Synchronization of unstable orbits using adaptive control

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We propose a method of controlling nonlinear and chaotic systems which is able to synchronize the phase space trajectory to a desired unstable orbit. The desired orbit could be an unstable periodic orbit or a chaotic orbit. The method uses the procedure of adaptive control and introduces time dependent changes in the system parameters. The changes in the parameter values depend on the deviations of the variables of the system from the desired orbit and the deviations of the controlled parameters from their values corresponding to the desired orbit. We illustrate our method using the Lorenz and Rössler systems. We also show that our method may be useful for communication purposes.

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I. INTRODUCTION

The existence of chaos in natural and man made systems is a well established fact. Recently, there has been considerable interest in controlling chaotic and nonlinear systems. Control of these systems is difficult because in these systems the natural tendency of nearby trajectories is to diverge exponentially in phase space, due to the sensitive dependence on initial conditions. On occasions chaos could be beneficial because it enhances the richness of the dynamical behavior. Hence, in such situations the interest may be in making the system follow a particular chaotic trajectory or the trajectory of a coevolving chaotic system. On the other hand, in some situations it is desired that chaos be eliminated and the system follows a time periodic trajectory. Different methods have been proposed to achieve these goals [1–22].

Ott, Grebogi, and Yorke [1,2] have proposed a method to convert the motion on a chaotic attractor to a desired periodic motion. For the desired periodic motion they choose a natural unstable periodic orbit of the system. This choice of the desired orbit has several advantages [1]. One waits for the trajectory of the system to come sufficiently close to the desired one and then the desired unstable periodic orbit is stabilized by making small time dependent perturbations of some set of available system parameters.

Pecora and Carroll [15,16] have shown that it is possible to synchronize two chaotic signals. They demonstrated that for a certain class of chaotic systems, two systems can be synchronized by dividing them each into two subsystems, namely, a drive subsystem and a response subsystem and keeping the variable values of the drive subsystem the same for both of them. Synchronization is obtained provided the subsystem Lyapunov exponents for the response subsystem are all negative. It has also been shown that [18] a dynamic feedback in the variable of the drive subsystem can also achieve synchronization. A possible application of synchronization of chaotic signals for communication purposes was demonstrated recently [23–25].

An alternative approach to the problem of synchronization of two chaotic systems was demonstrated recently by Lai and Grebogi [19]. The method is derived from the Ott, Grebogi, and Yorke control algorithm [1,2] and synchronization is achieved by introducing small perturbations of the parameters. However, for the successful implementation of this method one has to acquire a good amount of knowledge about the underlying attractor, by observing the system for a long time. Also, one must wait for the two trajectories to come sufficiently close to each other before perturbation can be applied.

Huberman and Lumer [20] introduced a simple adaptive control mechanism to control nonlinear systems. A system which is perturbed away from its stable fixed point value due to sudden deviations in parameter values is brought back to the fixed point by introducing appropriate changes in the parameter values. The method has been shown to be successful in the case of stable limit cycles also [21].

In this paper, we introduce a method for synchronizing the evolution of a nonlinear and chaotic system to a desired unstable trajectory through adaptive control. The desired unstable trajectory could be a chaotic orbit or an unstable periodic orbit. Our method is based on the idea of adaptive control suggested by Huberman and Lumer [20]. However, it should be noted that our method is capable of stabilizing an unstable orbit while the control suggested by Huberman and Lumer [20] works only for stable orbits. We assume that one or more of the system parameters are available for control and the values of these parameters for the desired orbit are known. The controlled parameters are changed depending on two factors; (1) the difference between the systems output variables and the corresponding variables of the desired orbit, and (2) the difference between the values of the parameters which are controlled and their values for the desired orbit. Since our approach is to introduce changes in the available system parameters, all the variables of the system evolve freely. Also, a detailed knowledge of the underlying attractor of the dynamics is not necessary. This avoids the need to observe the sys-

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tem for a long time before it is controlled.

In the following sections we give a description of our method of control (Sec. II) and demonstrate it using the well known Lorenz and Rössler systems (Sec. III). We also show that the method could be used for communication purposes (Sec. IV).

II. ADAPTIVE CONTROL FOR UNSTABLE ORBITS

We give a general description of our control procedure for flows. Consider an autonomous n-dimensional system evolving via the evolution equations

$$\dot{u}=f(u,\mu)\ , \tag{1}$$

where $u=(u_1,\ldots,u_n)$ and $f(u,\mu)=(f_1(u,\mu),\ldots,f_n(u,\mu))$ are n-dimensional vectors and the function f depends on the set of parameters $\mu=(\mu_1,\ldots,\mu_k)$. The values of the parameters μ are such that the system is in the chaotic regime. We assume that a set of parameters μ_i are available for control. Let $O(v)$ be the desired orbit and is a natural unstable orbit of the system and v denote the values of the variables of the desired orbit. The desired orbit may be a chaotic trajectory of a coevolving system or an unstable periodic orbit. Let the system generating the desired orbit be called the target system and the controlled system be called the response system. Our objective is to introduce a control mechanism for the system of Eq. (1) so that the variable u of the response system synchronize with the variables v of the target system. Keeping this in mind we modify the evolution of the system Eq. (1) by introducing small perturbations in the parameters μ_i;

$$\begin{aligned}&\dot{u}=f(u,\mu),\\&\dot{\mu}_i=-\epsilon h\left[(u_j-v_j),\operatorname{sgn}\left[\frac{df_j}{d\mu_i}\right]\right]-\delta g(\mu_i-\mu_i^*)\ ,\end{aligned} \tag{2}$$

where μ_i^* is the value of the parameters μ_i corresponding to the target system, ϵ is the stiffness constant, δ is the damping constant, and u_j denotes the variable in whose evolution equation the parameter μ_i occurs. The function h is a continuous function of the difference (u_j-v_j) and g is a continuous function of the difference $(\mu_i-\mu_j^*)$. The function $\operatorname{sgn}(x)$ denotes the sign of x. Parameters other than μ_i are assumed to be constant and set at the values of the target system. In our numerical examples we take

$$h\left[(u_j-v_j),\operatorname{sgn}\left[\frac{df_j}{d\mu_i}\right]\right]=\operatorname{sgn}\left[\frac{df_j}{d\mu_i}\right](u_j-v_j)\ , \tag{3}$$

$$g(\mu_i-\mu_i^*)=\mu_i-\mu_i^*\ . \tag{4}$$

Other forms of functions h and g are also possible.

In the absence of the control the system u will in general show a chaotic behavior. With control we find that the system is forced onto the desired orbit $O(v)$ of the target system, for certain ranges of values of the constants ϵ and δ. The range of values for which control is possible is determined by studying the Lyapunov characteristic exponents (LCE's) of the evolution Eqs. (2). The condition for synchronization with the desired trajectory is that all the LCE's are negative. The critical values of the stiffness constant and damping constant can be determined by the condition that the largest Lyapunov exponent be zero.

III. EXAMPLES

In this section we illustrate our control procedure using the Lorenz and Rössler systems.

A. Lorenz system

The Lorenz equations are given by [26]

$$\begin{aligned}&\dot{x}=-\sigma x+\sigma y\ ,\\&\dot{y}=rx-y-xz\ ,\\&\dot{z}=-bz+xy\ .\end{aligned} \tag{5}$$

We first consider the interesting case of synchronization with a desired chaotic orbit. The desired orbit may be predetermined or obtained from a coevolving system. We take the control parameter to be r. The equations for the parameter r is

$$\dot{r}=-\epsilon(y-y_c)\operatorname{sgn}\left[\frac{d\dot{y}}{dr}\right]-\delta(r-r^*)\ , \tag{6}$$

where y_c is the y component of the desired chaotic orbit

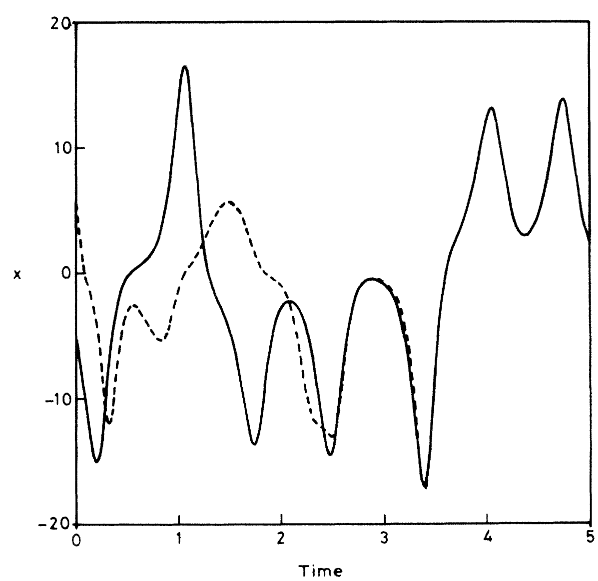

FIG. 1. The x variable of the target system and the response system are shown for the case of synchronization of two Lorenz systems evolving in the chaotic regime. The orbit generated by the target system is shown by the thick line and the orbit of the response system is shown by the dashed line. The target system parameters are $\sigma^*=10$, $r^*=28$, $b^*=\frac{8}{3}$. The response system has the same values for σ and b. Control is applied for r with stiffness constant $\epsilon=50$ and damping factor $\delta=20$. From the figure it is clear that the response system synchronizes with the target system within a short time and the two systems remain synchronized thereafter.

and r^* is the value of r corresponding to it. The values for the parameters σ and b remain the same as for the desired orbit. Our numerical investigations show that it is possible to synchronize the system with the desired chaotic orbit even when the system is started considerably away from the desired orbit.

Figure 1 shows the evolution of the x variable of the controlled Lorenz system and the desired chaotic orbit. The controlled system is started from a point away from the desired orbit. We see that the controlled system synchronizes with the desired chaotic orbit after some time. In Fig. 2 we plot the deviations of all the variables x, y, and z and the parameter r from the desired orbit values as a function of time for the case in Fig. 1. We find that the deviations decrease with time and the trajectory of the system synchronizes with the desired orbit. Also the parameter r tends to the value r^* of the desired orbit. Figure 3 shows the average transient time $\bar{\tau}$ as a function of δ, for a fixed value of ϵ, where the average transient time

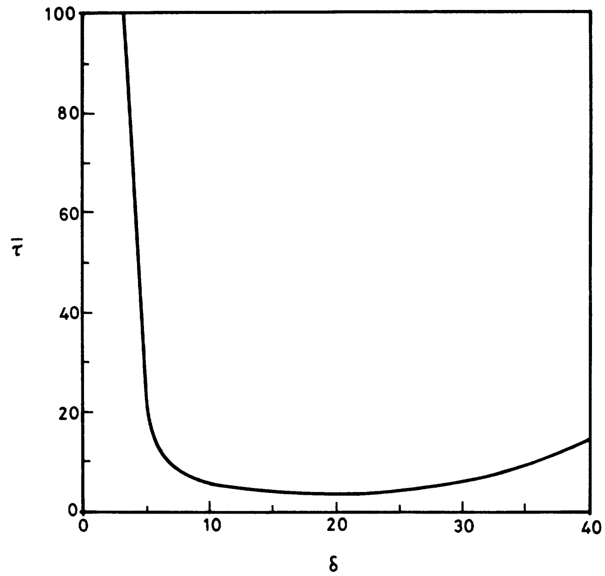

FIG. 3. The average transient time is plotted as a function of δ for $\epsilon=50$ for synchronizing the trajectory of the response Lorenz system with the chaotic orbit of the target Lorenz system. The system parameters are $\sigma^*=10$, $r^*=28$, $b^*=\frac{8}{3}$. The accuracy to which convergence is checked is 10^{-3} and the average was taken over 100 random initial conditions.

is obtained by considering the time taken for a trajectory starting from a random initial point to synchronize to the desired orbit within a given accuracy and averaging over several such random initial conditions. We find that the lower critical value of $\delta \approx 1.9$ below which synchronization is not possible. In Fig. 4 we plot the lower critical value of δ as a function of ϵ. The critical value of δ is obtained by calculating the largest Lyapunov exponent [28]

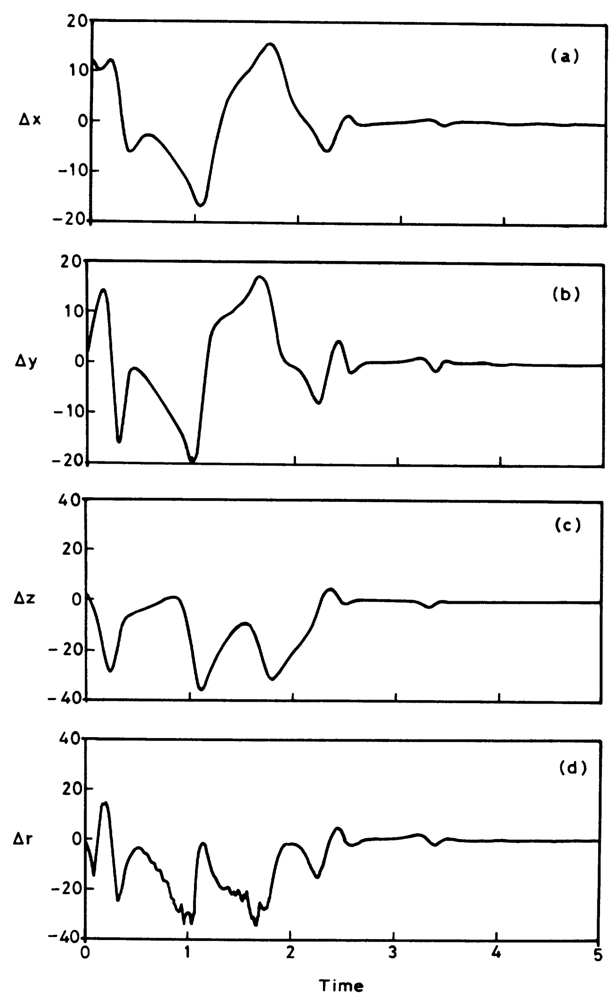

FIG. 2. The deviations $\Delta x, \Delta y, \Delta z$ of the three components of the chaotic trajectory of the response system from the target chaotic trajectory for the Lorenz systems of Fig. 1 shown as a function of time in figures (a), (b), and (c), respectively. The change Δr of the controlled parameter r from its value for the target trajectory r^* is shown in (d).

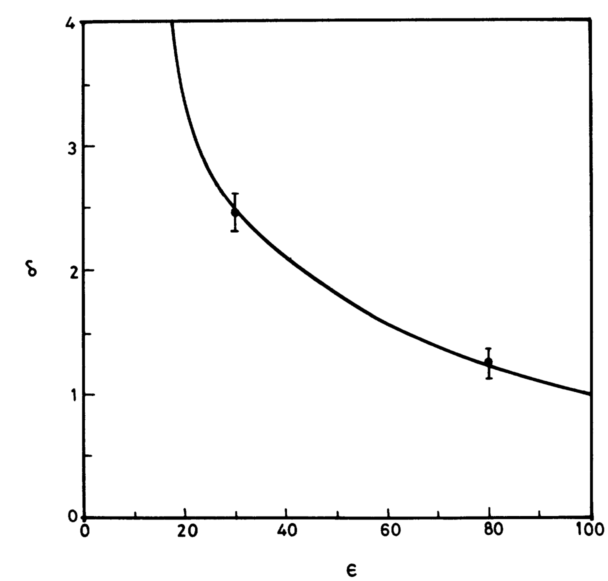

FIG. 4. The lower critical value of δ is plotted as a function of ϵ for synchronization of Lorenz system to a chaotic orbit. The system parameters are $\sigma^*=10$, $r^*=28$, $b^*=\frac{8}{3}$. Only two points are shown to indicate the error bar.

of the controlled chaotic trajectory and requiring that the largest Lyapunov exponent be zero at the critical value.

It is also possible to use the parameters σ and b for controlling the Lorenz system.

Let us next consider controlling the Lorenz Eq. (5) to focus the trajectory to an unstable periodic orbit. We illustrate the control using the case of the simplest periodic orbit, i.e., the fixed point. The Lorenz equations have three fixed points, $(0,0,0), [\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1]$. As an illustration consider the unstable fixed point $[\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1]$. The equation for the parameter r is

$$\dot{r} = -\epsilon(y-y^*)\operatorname{sgn}\left[\frac{d\dot{y}}{dr}\right] - \delta(r-r^*) , \tag{7}$$

where $y^* = \sqrt{b^*(r^*-1)}$. We write the Jacobian for Eqs. (5) and (7) as,

$$J = \begin{bmatrix} -\sigma & \sigma & 0 & 0 \\ r-z & -1 & -x & x \\ y & x & -b & 0 \\ 0 & -\epsilon & 0 & -\delta \end{bmatrix} . \tag{8}$$

The condition for control or focusing the trajectory on the fixed point is that the real part of all the eigenvalues of the matrix [Eq. (8)] at the fixed point should be negative. From this condition one can get the range of values of ϵ and δ for which control is possible. Figure 5 shows the x-y projection of a trajectory of the Lorenz system with control for r. The system evolves from an arbitrary initial condition away from the fixed point.

B. Rössler System

The Rössler equations are given by [27],

$$\begin{aligned} \dot{x} &= -y-z , \\ \dot{y} &= x+ay , \\ \dot{z} &= b+z(x-c) . \end{aligned} \tag{9}$$

We have applied our control technique to synchronize the trajectories of the Rössler system with desired chaotic orbits. It is possible to apply our control method by introducing changes in the parameter a. The equation for parameter a is

$$\dot{a} = -\epsilon(y-y_c)\operatorname{sgn}\left[\frac{d\dot{y}}{da}\right] - \delta(a-a^*) , \tag{10}$$

where y_c is the y component of the desired chaotic orbit and the evolution of the x, y, and z variables remain the same as in Eq. (9). Our numerical investigations showed that it is possible to synchronize the system with a desired chaotic orbit range of values for the constants ϵ and δ. Figure 6 shows the average transient time $\bar{\tau}$ as a function of δ for a particular value of ϵ. The lower critical value of δ is about 0.2. We again see that the average transient time diverges to infinity as the value of δ approaches the critical value.

We have also tried to control the system using other parameters of the system. However, we find that it is not possible to control the Rössler system by using the parameters b and c. This is contrary to the Lorenz system where control is possible using any of the three parameters σ, r, and b.

An analysis similar to that in the case of the Lorenz system applied for controlling the system to the unstable

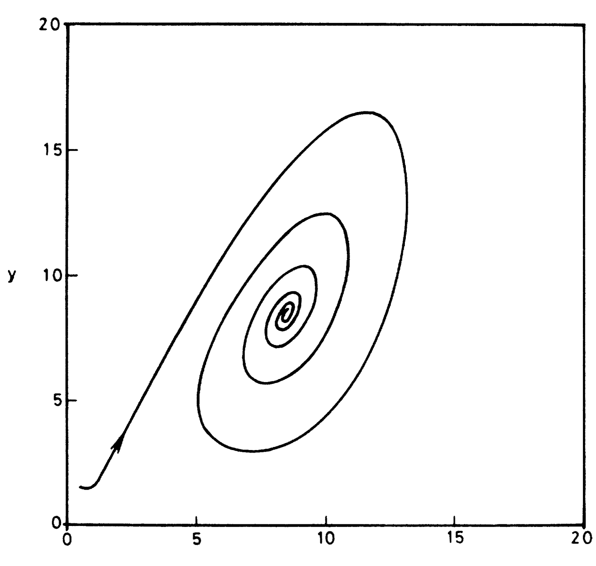

FIG. 5. The figure shows the x-y projection of a trajectory of the Lorenz system converging to the fixed point $x^* = y^* = \sqrt{b(r-1)}, z^* = r-1$ starting from an initial value away from the fixed point. Control is applied for the parameter r. The system parameters are $\sigma^* = 10$, $r^* = 28$, $b^* = \frac{8}{3}$. The values of the constants are $\epsilon = 10$ and $\delta = 10$.

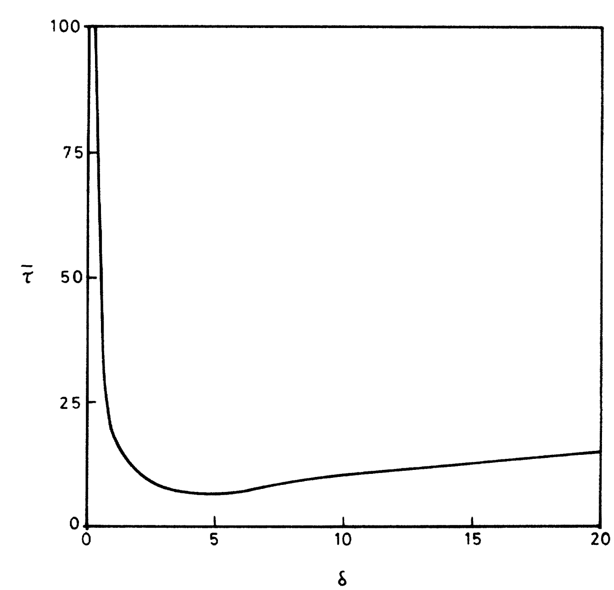

FIG. 6. The figure shows the average transient time as a function of δ for the Rössler system for synchronization with a chaotic orbit. The parameter values are $a^* = 0.398$, $b^* = 2.0$, and $c^* = 4.0$ and the stiffness constant $\epsilon = 10$.

fixed points. Again control is possible only with the parameter a.

IV. APPLICATION TO COMMUNICATION

Recently, it was shown that the combination of synchronization and unpredictability from purely deterministic systems leads to some potentially interesting communication applications [23]. Here we show that our method of control could be used for communication purposes.

The transmitter and the receiver are systems with identical parameter ranges. A chaotic signal sent from the transmitter is received by the receiver and is used to calculate the perturbations of one of the parameters which will ultimately synchronize the receiver with the transmitter. In this process the controlled parameter also gets synchronized with that of the transmitter. Now if the parameter of the transmitter is modulated by some signal, it will change the chaotic signal being sent. The modulating signal could be deciphered from the controlled parameter values at the receiver. Figure 7 shows this process of communication for a binary signal using the Lorenz system. The modulation of the transmitter signal is done by varying the parameter r between the values 28 and 28.5 corresponding to the binary 0 and 1 values of a square wave. The signal being transmitted is the y variable of the Lorenz system. The receiver system is controlled using parameter r with $r^*=28$ [Eq. (6)]. The plot of the parameter r at the receiver clearly shows a spike corresponding to a binary 1. The beginning of the spike corresponds to the beginning of the binary 1 of the transmitted signal, but the end of the binary 1 transmission could not be determined accurately. One is able to decipher the message provided the time delay between the spikes is well above the time required for synchronization within a specified accuracy.

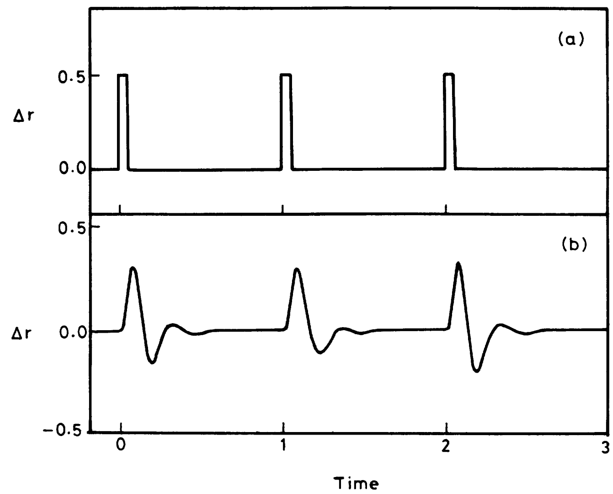

FIG. 7. The figure shows the process of communicating a binary valued bit stream using adaptive control. The transmitter and the receiver are two Lorenz systems. The system parameters are $\sigma^*=10$, $r^*=28$, $b^*=\frac{8}{3}$, $\epsilon=100$, and $\delta=20$. A portion of the binary bit stream, which is modulating the parameter r of the transmitter is shown in (a) Δr denotes the difference of r from r^*. Each binary 1 state lasts for a time of 0.05 and each binary 0 state lasts for a time of 0.95. The variation of r in the receiver system is shown in (b). Corresponding to each binary 1 in (a), there is a spike in (b).

V. DISCUSSION

We have shown that it is possible to control nonlinear and chaotic systems by applying a control in a set of available system parameters. A system parameter is perturbed depending on the deviation of a system variable from the desired value and also on the deviation of the parameter from its correct value corresponding to the desired trajectory. The technique can be used to synchronize the chaotic trajectory of a system to that of a target system. The method could as well be used to control the system to an unstable fixed point or an unstable periodic orbit. In our calculations, we have considered those variables in whose equations the controlled parameters appear. But this need not be a necessary condition for control. In practice, it may be possible to use any of the available variables [21].

We have also shown that it is possible to communicate a binary message using our control method. The communication method using the synchronization procedure of Pecora and Carroll requires control of the variables of the drive subsystem. On the other hand, our method utilizes perturbations in system parameters. Thus, our method should be useful when the parameters can be controlled but not the variables.

ACKNOWLEDGMENT

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